Critical discussion of the two-loop calculations for the Kardar-Parisi-Zhang equation

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(Received 16 June 1997)

In this paper, we perform a careful analysis of the renormalization procedure used in existing calculations to derive critical exponents for the Kardar-Parisi-Zhang equation at two-loop order. This analysis explains the discrepancies between the results of the different groups. The correct critical exponents in $d=2+\varepsilon$ dimensions at the crossover between the weak- and strong-coupling regime are $\chi=O(\varepsilon^3)$ and $z=2+O(\varepsilon^3)$. No strong-coupling fixed point exists at two-loop order. [S1063-651X(97)08010-0]

PACS number(s): 05.40.+j

I. INTRODUCTION

The Kardar-Parisi-Zhang (KPZ) equation [1]

$$\frac{\partial h(x,t)}{\partial t}=\nu\nabla^2 h(x,t)+\frac{\lambda}{2}[\nabla h(x,t)]^2+\eta(x,t), \tag{1}$$

$$\overline{\eta(x,t)\,\eta(x',t')}=2D\,\delta^d(x-x')\,\delta(t-t') \tag{2}$$

plays a central role as the simplest field-theoretic model for nonlinear growth and has extensively been studied during the last years. Whereas the situation is clear for space dimension $d=1$, the (physically more interesting) case of $d\geqslant 2$ can only be attacked by approximative methods or field-theoretic perturbative expansions. Using the latter, the fixed point structure of the renormalization-group flow for $d=2+\varepsilon$ has been obtained. Two domains can be distinguished: For small effective coupling

$$g=\frac{\lambda^2 D}{\nu^3}, \tag{3}$$

the renormalization group flow tends to 0 in the long-wavelength limit, for large coupling to a strong-coupling fixed point $g=g_{\mathrm{SC}}$. The crossover takes place at $g=g_{\mathrm{CO}}$, which turns out to be of order ε in an ε expansion and can therefore be studied perturbatively.

Three such calculations have been performed, one by Sun and Plischke [2], another by Teodorovich [3], the third by Frey and Täuber [4]. The results are contradictory. We were able to trace back the difference to a problem in the application of the renormalization-group procedure, which led [2] and [3] to incorrect results, see the discussion in Secs. III, IV, and V. We will first derive some elementary and hopefully well-known properties of the renormalization group, which will help us to make the point clear.

II. SOME ELEMENTARY PROPERTIES ABOUT RENORMALIZATION AND THE FREEDOM TO CHOOSE A RENORMALIZATION SCHEME

In the following, we want to exploit the freedom one has to choose renormalization factors in the dimensional regularization scheme. As an example, take any massless theory and suppose that the derivative with respect to k^2 of the bare and renormalized two-point functions (we denote this by a dot) at a given renormalization scale μ are related by

$$\dot{\Gamma}^{(2)}_B=Z(g_R)\dot{\Gamma}^{(2)}_R. \tag{4}$$

The factor $Z(g_R)$, which is a function of the renormalized coupling g_R and ε only, is introduced to define a finite renormalized two-point function $\Gamma^{(2)}_R$. Then, it is standard to obtain the anomalous dimension of the field, given by the renormalization-group ζ function, see, e.g., [5], as variation of $\ln(Z)$ with respect to the renormalization scale μ, keeping bare quantities fixed,

$$\zeta(g_R)=-\frac{1}{2}\mu\frac{\partial}{\partial\mu}\bigg|_B \ln Z(g_R)=-\frac{1}{2}\beta(g_R)\,\frac{\partial}{\partial g_R}\ln Z(g_R). \tag{5}$$

In the second line, we have introduced the renormalization-group β function, which is obtained from the variation of the renormalized coupling g_R with respect to the renormalization scale μ

$$\beta(g_R)=\mu\frac{\partial}{\partial\mu}\bigg|_B g_R, \tag{6}$$

and where renormalized and bare coupling constants g_R and g_B are related by

$$g_B=\mu^{\varepsilon}Z_g^{-1}(g_R)g_R. \tag{7}$$

The critical point g^* is the zero of the β function

$$\beta(g^*)=0. \tag{8}$$

The critical exponent ζ^* is

$$\zeta^*=\zeta(g^*). \tag{9}$$

All these definitions are standard [5] and are used (in slightly different notations) in [2–4]. ([3] uses a cutoff regularization, but there is no principal difference.) We now ask the question: Is the renormalization factor $Z(g_R)$, which was introduced in Eq. (4), unique? The answer is no. $Z(g_R)$ can be multiplied by any finite factor $Z_f(g_R)$ and $\Gamma^{(2)}_R$ remains finite. Instead of $Z(g_R)$ one could therefore use $Z'(g_R)$,

$$Z'(g_R)=Z(g_R)Z_f(g_R). \tag{10}$$

Let us verify that this is consistent and that the critical exponent ζ^* is unchanged. The new ζ' function is

$$\zeta'(g_R)=-\frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z'(g_R)$$

$$=\zeta(g_R)-\frac{1}{2}\beta(g_R)\,\frac{\partial}{\partial g_R}\ln Z_f(g_R). \tag{11}$$

As $Z_f(g_R)$ is finite for any g_R, $\beta(g_R)(\partial/\partial g_R)\ln Z_f(g_R)$ tends to 0 for $g_R\to g^*$ and we obtain the result

$$\zeta'(g^*)=\zeta(g^*). \tag{12}$$

One can also prove that the freedom to choose g_R is a simple finite reparametrization

$$g_R'=g_R Z_f(g_R). \tag{13}$$

One has, however, to be careful to make this change of variables in every renormalization factor.

Let us now discuss the minimal subtraction scheme. Suppose that $\Gamma_B^{(2)}$ at the renormalization point μ satisfies up to two-loop order, i.e., up to order g_R^2,

$$\dot\Gamma_B^{(2)}=\left(1+a_{-1}\frac{g_R}{\varepsilon}+a_0 g_R+a_1 g_R\varepsilon+\cdots+b_{-2}\frac{g_R^2}{\varepsilon^2}+b_{-1}\frac{g_R^2}{\varepsilon}\right.$$

$$\left.+b_0 g_R^2+b_1 g_R^2\varepsilon+\cdots\right)1, \tag{14}$$

where a_i and b_i are constants and where we have suppressed higher order terms in ε. We now want to introduce renormalization factors. Of course, setting in Eq. (4)

$$Z(g_R)=1+a_{-1}\frac{g_R}{\varepsilon}+a_0 g_R+a_1 g_R\varepsilon+\cdots+b_{-2}\frac{g_R^2}{\varepsilon^2}+b_{-1}\frac{g_R^2}{\varepsilon}$$

$$+b_0 g_R^2+b_1 g_R^2\varepsilon+\cdots, \tag{15}$$

makes $\Gamma_R^{(2)}$ finite. [The dots stand for the same higher order terms as in Eq. (14)]. But this is not the simplest possible choice. The idea of the minimal subtraction scheme is now to choose a Z factor, which contains only the pole terms in ε. One is therefore tempted to replace Z by

$$Z_{\rm PM}(g_R)=1+a_{-1}\frac{g_R}{\varepsilon}+b_{-2}\frac{g_R^2}{\varepsilon^2}+b_{-1}\frac{g_R^2}{\varepsilon}. \tag{16}$$

The reader is invited to stop for a second to think about this choice before going on to read.

Let us see whether $Z_{\rm PM}$ correctly renormalizes the theory. Then, the ratio of Z and $Z_{\rm PM}$ has to be finite. Expanding up to order g_R^2 and dropping all terms of order ε^0 and higher, we obtain

$$\frac{Z(g_R)}{Z_{\rm PM}(g_R)}=1-a_0a_{-1}\frac{g_R^2}{\varepsilon}+\cdots. \tag{17}$$

This ratio is not finite and $Z_{\rm PM}$ therefore not a correct renormalization factor. We will henceforth call this wrong prescription to find the Z factor in the minimal subtraction scheme the ''pseudo-minimal-subtraction scheme'' in order to make clear that this very special mistake was made.

The correct factor to be used in a minimal-subtraction scheme would be

$$Z_{\rm MS}(g_R)=1+a_{-1}\frac{g_R}{\varepsilon}+b_{-2}\frac{g_R^2}{\varepsilon^2}+(b_{-1}-a_0a_{-1})\,\frac{g_R^2}{\varepsilon} \tag{18}$$

as the reader may easily convince himself by the construction given above.

Apparently, the term a_0 cannot be neglected at two-loop order. Let us determine more generally which terms may intervene in an n-loop calculation. First of all, the perturbative fixed point in an ε expansion is of order ε

$$g^*\sim\varepsilon. \tag{19}$$

(There are situations where g^* is of order $\sqrt{\varepsilon}$ or similar. For those cases the following argument has to be modified.) Let us now determine in which order of ε a term of order

$$\frac{g_R^k}{\varepsilon^l} \tag{20}$$

in $Z(g_R)$ will contribute to the ζ function at $g_R=g^*$. Taking care of the factor ε, which comes from the variation of $\ln Z(g_R)$ in the definition of $\zeta(g_R)$, Eq. (5), the answer is, with the help of Eq. (19)

$$\varepsilon\,\frac{(g^*)^k}{\varepsilon^l}=\varepsilon^{1+k-l}. \tag{21}$$

To calculate the critical exponent ζ^* up to order ε^n, one therefore has to determine all contributions to Z which are proportional to g_R^k up to order ε^{n-k-1}.

This means that to calculate up to order ε^2, one can drop terms from Z and retain only

$$Z_{\rm d}(g_R)=1+a_{-1}\frac{g_R}{\varepsilon}+a_0 g_R+b_{-2}\frac{g_R^2}{\varepsilon^2}+b_{-1}\frac{g_R^2}{\varepsilon}. \tag{22}$$

Note again that a_0 cannot be dropped.

In the next sections we will apply these considerations to the two-loop calculations of the KPZ equation in the literature.

III. ANALYSIS OF THE WORK BY SUN AND PLISCHKE

In this section we are going to analyze the two-loop renormalization-group calculations by Sun and Plischke [2], see also the comment in [6]. From their Appendixes A and B it is clear that the finite part of the one-loop contribution is dropped. They write ''In this Appendix we list all expressions of Feynman graphs ... and their final results in $2-\varepsilon$ dimensions to order $O(\varepsilon^{-1})$.'' As there is a finite contribution at one-loop order, the discussion given above shows that their results are incorrect. We would have liked to incorpo-

rate the missing term in their calculations but as not all Z factors are given explicitly we have to leave it to the authors.

IV. ANALYSIS OF THE WORK BY TEODOROVICH

Let us also look at the article by Teodorovich [3]. Teodorovich is conscious that the finite part of the one-loop diagrams enters as insertion of these diagrams into two-loop diagrams and he explicitly calculates the finite contributions to the one-loop diagrams in his equations (5.1) and (5.2). But then he also drops the finite contribution to Σ^R, which is still present in Eq. (5.1), in Eq. (5.3). We are therefore led to the same conclusion as in the case of Sun and Plischke: The results are incorrect. Let us, however, mention that Teodorovich does not agree with our criticism. He states that he used the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) procedure and inserted one-loop renormalized terms in order to calculate the renormalization factors [8]. It, is, however, impossible from his article [3] to get the precise procedure enabling one to verify his claim.

V. ANALYSIS OF THE WORK BY FREY AND TÄUBER

In this section we are going to analyze the two-loop renormalization-group calculations by Frey and Täuber [4], see also their comment in [7].

They introduce the following two renormalization constants for the renormalization of D and ν, see [4], Eqs. (3.43) and (3.44):

$$Z_D=1-\frac{\hat{g}_B}{\varepsilon}-(d-1)\frac{\hat{g}_B^2}{\varepsilon}-\frac{d-2}{d}\frac{\hat{g}_B^2}{2\varepsilon}+(d-1)\frac{\hat{g}_B^2}{\varepsilon^2}+O(\hat{g}_B^3), \tag{23}$$

$$Z_\nu=1+\frac{d-2}{d}\left[\frac{\hat{g}_B}{\varepsilon}+(d-1)\frac{\hat{g}_B^2}{2\varepsilon}+\frac{d-2}{d}\frac{\hat{g}_B^2}{2\varepsilon}-(d-1)\frac{\hat{g}_B^2}{2\varepsilon^2}\right] -\frac{d-1}{d}\frac{\hat{g}_B^2}{2\varepsilon}+O(\hat{g}_B^3), \tag{24}$$

$$\hat{g}_B=\mu^\varepsilon g_B, \tag{25}$$

$$\varepsilon=d-2. \tag{26}$$

(Note that our $\hat{g}_B=\hat{g}_0$ in the notation of [4]. To simplify the considerations, we also have replaced the function $F_\nu(d)$ in Eq. (3.44) of [4] by its value at $d=2$, see Eq. (A38) of [4]. The difference is a term of order ε^0 and can be neglected in two-loop order.) As we will check later these Z factors subtract all divergences correctly. Let us first explain the appearance of factors d and ε. Frey and Täuber [4] use a dimensional regularization prescription, where divergences appear as poles in $1/\varepsilon$. This is the origin of the factors $1/\varepsilon$ and $1/\varepsilon^2$ in Eqs. (23) and (24). On the other hand, there are integrals of the type

$$\int_k [2(\vec{k}\cdot\vec{p})^2-k^2p^2]f(k^2), \tag{27}$$

which are identical to

$$p^2\frac{2-d}{d}\int_k k^2 f(k^2). \tag{28}$$

Frey and Täuber call the factors $(d-2)$ ''geometrical'' because they result from scalar products in the integrand. The renormalization scheme in Ref. [4] is such that those geometrical terms are not ε expanded in order to meet the condition that a fluctuation-dissipation theorem is valid in $d=1$ dimensions. This procedure results in the appearance of the first nontrivial term in Z_ν [see Eq. (24)],

$$\frac{d-2}{d}\frac{\hat{g}_B}{\varepsilon}. \tag{29}$$

From the reasoning in [4] one could gain the impression that such a term can be dropped, if the distinction between factors $d-2$ and ε is not made. This would be incorrect, since—as we have explained in detail in Sec. II—this term has to be retained in order to renormalize the theory. We would write Eq. (29) as

$$\left(\frac{1}{2}+O(\varepsilon)\right)\hat{g}_B \tag{30}$$

and using our findings at the end of Sec. II conclude that the term of order ε^0 has to be taken into account. [To be precise: In Sec. II we had argued on the level of the Z factors as functions of the renormalized coupling g_R. As $g_B=g_R+O(g_R^2)$ this does not affect our reasoning for the one-loop term in Eqs. (29) and (30).]

This analysis shows that the distinction between factors $d-2$ and ε is not necessary in order to renormalize the theory above $d=2$ dimensions. Especially, one may not misinterpret [4] in the way that one could choose d fixed in the so-called ''geometrical'' factors and then perform the limit $\varepsilon\to 0$ elsewhere. This would result in a divergent term of order $1/\varepsilon$ in the renormalization-group functions, violating the very principle of renormalizability.

The correct procedure to follow is the following: Expand the Z factors in ε only. Retain terms of order $\hat{g}_B$, $\hat{g}_B/\varepsilon$, $\hat{g}_B^2/\varepsilon$, and $\hat{g}_B^2/\varepsilon^2$. (Of course, there would be no harm in keeping higher order terms in ε, but this is unnecessary.) Then calculate the renormalization-group functions.

We refrain from discussing the validity of a perturbative renormalization-group approach below the lower critical dimension $d=2$. Such a scheme would have to meet certain criteria, such as the validity of the fluctuation-dissipation theorem in $d=1$. But, it is not clear to us how this procedure should be defined.

In order to demonstrate the correctness of our procedure and in order to show that factors $d-2$ and ε do not have to be distinguished, we use the Z factors of [4] in our scheme to establish the critical exponents. In view of our discussion in Sec. II we first check that the finite term in the factor Z_D is correctly taken into account. This is trivially true as normalizations in [4] are chosen such that the only integral which has to be calculated in one-loop order gives exactly $1/\varepsilon$ and *no* finite contribution.

Replacing $d=2+\varepsilon$ and retaining only terms of order $\hat{g}_B$, $\hat{g}_B/\varepsilon$, $\hat{g}_B^2/\varepsilon$, and $\hat{g}_B^2/\varepsilon^2$, the Z factors are

$$Z_D(\hat{g}_B)=1-\frac{\hat{g}_B}{\varepsilon}+\frac{\hat{g}_B^2}{\varepsilon^2}, \tag{31}$$

$$Z_\nu(\hat{g}_B)=1+\frac{1}{2}\,\hat{g}_B-\frac{1}{2}\,\frac{\hat{g}_B^2}{\varepsilon}, \tag{32}$$

$$Z_g(\hat{g}_B)=1-\left(\frac{3}{2}+\frac{1}{\varepsilon}\right)\hat{g}_B+\left(\frac{3}{\varepsilon}+\frac{1}{\varepsilon^2}\right)\hat{g}_B^2. \tag{33}$$

The last factor, Z_g, was calculated via the relation

$$Z_g=\frac{Z_D}{Z_\nu^3} \tag{34}$$

as stated in [4]. It is now necessary to transform to renormalized quantities. Replacing $\hat{g}_B$ in Eqs. (31)–(33) by (note that this is sufficient in two-loop order)

$$\hat{g}_B=g_R+\left(\frac{1}{\varepsilon}+\frac{3}{2}\right)g_R^2+O(g_R^3) \tag{35}$$

we obtain

$$Z_D(g_R)=1-\frac{g_R}{\varepsilon}-\frac{3}{2}\,\frac{{g_R}^2}{\varepsilon}, \tag{36}$$

$$Z_\nu(g_R)=1+\frac{1}{2}g_R, \tag{37}$$

$$Z_g(g_R)=1-\frac{3}{2}g_R-\frac{g_R}{\varepsilon}. \tag{38}$$

The renormalization-group functions as defined in [4] are

$$\zeta_D(g_R)=\mu\frac{\partial}{\partial\mu}\bigg|_B \ln Z_D(g_R)=\beta(g_R)\,\frac{\partial}{\partial g_R}\ln Z_D(g_R), \tag{39}$$

$$\zeta_\nu(g_R)=\mu\frac{\partial}{\partial\mu}\bigg|_B \ln Z_\nu(g_R)=\beta(g_R)\,\frac{\partial}{\partial g_R}\ln Z_\nu(g_R), \tag{40}$$

$$\begin{aligned}\beta(g_R)&=\mu\frac{\partial}{\partial\mu}\bigg|_B g_R\\ &=\frac{\varepsilon g_R}{1-g_R\dfrac{\partial}{\partial g_R}\ln Z_g(g_R)}\\ &=g_R[\varepsilon+\zeta_D(g_R)-3\zeta_\nu(g_R)].\end{aligned} \tag{41}$$

Using our results from above, we obtain the following two-loop renormalization-group functions:

$$\zeta_D(g_R)=-g_R-\frac{3}{2}g_R^2, \tag{42}$$

$$\zeta_\nu(g_R)=\frac{\varepsilon}{2}g_R-\frac{1}{2}g_R^2, \tag{43}$$

$$\beta(g_R)=\varepsilon g_R-\left(1+\frac{3}{2}\varepsilon\right)g_R^2, \tag{44}$$

which are identical to the results obtained in Eqs. (3.62)–(3.64) of [4] after a full ε expansion. There is one nontrivial ($g^*\neq 0$) zero of the β function. It is located at

$$g^*=\varepsilon-\frac{3}{2}\varepsilon^2+O(\varepsilon^3). \tag{45}$$

Note that no fixed point of order 1 (''nonperturbative fixed point'') exists in contrast to [2] and [3]. This yields the critical exponents

$$\zeta_D^*=-\varepsilon+O(\varepsilon^3), \tag{46}$$

$$\zeta_\nu^*=O(\varepsilon^3). \tag{47}$$

The more standard quantities, the roughness exponent χ and the dynamical exponent z, are

$$\chi=O(\varepsilon^3), \tag{48}$$

$$z=2+O(\varepsilon^3). \tag{49}$$

In addition, the correction to scaling exponent

$$\omega=\frac{\partial}{\partial g_R}\beta(g_R)\big|_{g_R=g^*} \tag{50}$$

is easily read off from Eq. (44),

$$\omega=\varepsilon+O(\varepsilon^3). \tag{51}$$

We have given all calculations in this explicit form to convince the reader that the standard ε expansion is sufficient in order to obtain the critical exponents given in Eqs. (48) and (49).

For all proponents of the minimal-subtraction scheme, we leave it as an exercise that via a variable transformation and finite renormalizations one can transform to the following Z factors in the minimal-subtraction scheme:

$$\begin{aligned}Z_D(g_R)&=1-\frac{g_R}{\varepsilon}+O(g_R^3),\\ Z_\nu(g_R)&=1+O(g_R^3),\\ Z_g(g_R)&=1-\frac{g_R}{\varepsilon}+O(g_R^3).\end{aligned} \tag{52}$$

Using these Z factors, one of course obtains the same critical exponents to order ε^2.

We also found some minor misprints in [4]. For the reader's convenience we give a list here [9].

VI. CONCLUSIONS

In this article we have analyzed the two-loop renormalization-group calculations by Sun and Plischke [2], by Teodorovich [3], and by Frey and Täuber [4]. Sun and Plischke [2] and Teodorovich [3] do not correctly take into account the finite part of the one-loop diagrams. Therefore their results are incorrect.

The renormalization scheme by Frey and Täuber [4] leads to the correct result. From the reasoning given in their paper

one could, however, gain the wrong impression that a distinction between so-called geometrical factors and $1/\varepsilon$ poles is necessary in order to get a valid renormalization of the KPZ equation. We have shown here that such a distinction is in fact not necessary above the lower critical dimension $d=2$. This conclusion is supplemented through work by Lässig, who showed [10], using the mapping to directed polymers, that the predicted critical exponents

$$\zeta = 0, \qquad z = 2 \tag{53}$$

are correct to all orders in ε. This result had already been obtained in a different context in [11].

On the technical level, we have shown that the dimensional regularization scheme works perfectly well and does not have to be modified in order to calculate the critical exponents for the KPZ equation to order ε^2 above the lower critical dimension. The analysis of the critical behavior below $d=2$ is more subtle and it is presently not clear whether there is a valid concept for the analysis of the strong-coupling behavior.

ACKNOWLEDGMENTS

I thank H. W. Diehl, M. Kardar, J. Krug, L. Schäfer, and especially J. Hager for valuable discussions.

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[9] We found the following misprints in [4]: Eq. (2.17) reads $[\lambda_0]=\Lambda^{-d/2+1}\nu_0^{3/2}D_0^{-1/2}$, Eq. (3.28) reads $(\partial/\partial q^2)\Gamma_{\tilde{h}h}(\mathbf{q},\omega)^{\mathrm{sing}}|_{\mathbf{q}=0}=\cdots$, Eq. (3.48) reads $\beta_g=g\{d-2-[2(2d-3)/d]g-[(d-1)(5d-7)/d]g^2-3(d-1)/8d]g^2F_\nu(d)+O(g^3)\}$, and Eq. (A24) reads $\int_p(\mathbf{qp})^2f(p)=(q^2/d)\int_p p^2f(p)$.
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